DEFERRED CESÀRO MEAN OF DOUBLE SEQUENCES

Şeyda SEZGEK and İlhan DAĞADUR

Mersin University

2nd International Intuitionistic Fuzzy Sets Conference

14-18 October 2015

Abstract

In this study, the concepts of deferred Cesàro mean and deferred statistical convergence of double sequences are defined and studied by using deferred double natural density of the subset of natural numbers. Also, we obtain some inclusion results between Cesàro submethod $C_{\lambda,\mu}$ and deferred Cesàro mean $D_{\beta,\gamma}$ of the double sequences.

In 1932, Agnew [1] defined the deferred Cesàro mean $D_{p,q}$ of the sequence $x = (x_k)$ by

$$(D_{p,q}x)_n := \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of positive natural numbers satisfying

$$p(n) < q(n) \quad \text{and} \quad \lim_{n\to\infty} q(n) = \infty.$$

By the convergence of a double sequence we mean the convergence in Pringsheim's sense [15]. A double sequence $x = (x_{jk})$ is said to be convergent in the Pringsheim's sense if for all $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon)$ such that $|x_{nm} - L| < \varepsilon$ whenever $n, m \geq n_0$ [15]. In this case, we write $P - \lim_{j,k\to\infty} x_{jk} = L$.

A double sequence $x = (x_{jk})$ is bounded if there exist a positive number M such that $\left|x_{jk}\right| < M$ holds for all $(j, k) \in \mathbb{N} \times \mathbb{N} = \mathbb{N}^2$, i.e., if

$$\| x \|_{(\infty,2)} := \sup_{j,k} \left|x_{jk}\right| < \infty.$$

We will denote the set of all bounded double sequences by $l_{(\infty,2)}$. Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded.

In Mursaleen's study[11], let $K \subset \mathbb{N}^2$ be a two-dimensional set of positive integers and let

$$K(n,m) := \{(j,k) \in K \ : \ (j,k) \leqslant (n,m)\}.$$

Then, the lower asymptoic density of the set $K \subset \mathbb{N}^2$ is defined as

$$\underline{\delta}_2(K(n,m)) := \liminf_{n,m\to\infty} \frac{|K(n,m)|}{mn},$$

if the limit exists and finite. The vertical bars above indicate the cardinality of the set $K(n,m)$. In case the sequence $\left(\frac{|K(n,m)|}{mn}\right)$ has a limit in Pringsheim's sense then we say that K has a double natural density and is defined as

$$\delta_2(K(n,m)) := \lim_{n,m\to\infty} \frac{|K(n,m)|}{mn}.$$

In[11], a double sequence $x = (x_{jk})$ is statistically convergent to the number L if for each $\epsilon > 0$

$$\lim_{n,m\to\infty} \frac{1}{nm} \left| \{(j,k) : j \leq n,\ k \leq m, \mid x_{jk} - L \mid \geq \epsilon\} \right| = 0.$$

In this case, we write $st_2 - \lim_{j,k\to\infty} x_{jk} = L$ and we denote the set of all double statistically convergent sequences by st_2.
Let $A = (a_{jk}^{nm})$ be a four dimensional summability matrix and $x = (x_{jk})$ be a double sequence. If $[Ax] := \{(Ax)_{nm}\}$ is P-convergent to L then we say (x_{jk}) is $A-$summable to L where

$$(Ax)_{nm} := \sum_{j,k} a_{jk}^{nm} x_{jk} \quad for\ all\ \ n,m \in \mathbb{N}$$

A said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit [10]. Recall that four dimensional Cesàro matrix $C_1 = (c_{jk}^{nm})$ is defined by

$$c_{jk}^{nm} = \begin{cases} \frac{1}{nm}, & j \leq n \text{ and } k \leq m \\ 0, & \text{otherwise.} \end{cases}$$

Let the index sequences $\lambda(n)$ and $\mu(m)$ are strictly increasing single sequences of positive integers and $x = (x_{jk})$ be a double sequence. Then, the Cesàro submethod $C_{\lambda,\mu} := (C_{\lambda,\mu}, 1, 1)$ is defined to be

$$(C_{\lambda,\mu}x)_{nm} = \frac{1}{\lambda(n)\mu(m)} \sum_{j=1,k=1}^{\lambda(n),\mu(m)} x_{jk}$$

where $\sum_{j=1,k=1}^{\lambda(n),\mu(m)} x_{jk} = \sum_{j=1}^{\lambda(n)} \sum_{k=1}^{\mu(m)} x_{jk}$. Since $\{(C_{\lambda,\mu}x)_{nm}\}$ is a subsequence of $\{(C_1 x)_{nm}\}$, the method $C_{\lambda,\mu}$ is RH-regular for any λ, μ [19].

Definition

Let $x = (x_{kl})$ be a double sequence and $\beta(n) = q(n) - p(n)$, $\gamma(m) = r(m) - t(m)$,and let $\{p(n)\}$, $\{q(n)\}$, $\{r(m)\}$ and $\{t(m)\}$ are sequences of nonnegative integers satisfying the conditions

$$\begin{gathered} p(n) < q(n) \quad , \quad t(m) < r(m) \\ \text{and} \\ \lim_{n\to\infty} q(n) = \infty \quad , \quad \lim_{m\to\infty} r(m) = \infty. \end{gathered} \tag{1}$$

Then deferred Cesàro mean $D_{\beta,\gamma}$ of the double sequence x is defined by

$$(D_{\beta,\gamma}x)_{nm} = \frac{1}{\beta(n)\,\gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),\ r(m)} x_{kl}.$$

We note that $D_{\beta,\gamma}$ is clearly regular for any choice of $\{p(n)\}$, $\{q(n)\}$, $\{r(m)\}$ and $\{t(m)\}$. Throughout this paper $\beta(n) = q(n) - p(n)$, $\gamma(m) = r(m) - t(m)$ are represented β and γ respectively.

Definition

Let $x = (x_{kl})$ be a double sequence and a real number L. Then the double sequence x is said to be $D_{\beta,\gamma}$-summable to L if

$$\lim_{n,m\to\infty} \frac{1}{\beta(n)\,\gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),\ r(m)} (x_{kl} - L) = 0$$

exists and it is denoted by $(D_{\beta,\gamma}) - \lim_{n,m\to\infty} x_{nm} = L$ or $\lim_{n,m\to\infty}(D_{\beta,\gamma}x)_{nm} = L$.

Definition

Let K be a subset of $\mathbb{N}^2$ and denote the set

$$\{(k,l) : p(n) < k \leq q(n),\ t(m) < l \leq r(m),\ (k,\ l) \in K\}$$

by $K_{\beta,\gamma}(n,m)$. The deferred double natural density of K is defined by

$$\delta^{(2)}_{D_{\beta,\gamma}}(K) := \lim_{n,m\to\infty} \frac{1}{\beta(n)\ \gamma(m)} |K_{\beta,\gamma}(n,\ m)|$$

whenever the limit exists. The vertical bars indicate the cardinality of the set $K_{\beta,\gamma}(n,m)$.

Definition: A double sequence $x = (x_{kl})$ is said to be deferred statistically convergent to $L \in \mathbb{N}$ if for every $\varepsilon > 0$,

$$\lim_{n,m\to\infty} \frac{|\{(k,l) : p(n)+1 \leq k \leq q(n), t(m)+1 \leq l \leq r(m), |x_{kl} - L| \geq \varepsilon\}|}{\beta(n)\gamma(m)}$$

=0
and it is denoted by $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$.

Theorem

Let $\{p(n)\}$, $\{q(n)\}$, $\{r(m)\}$ *and* $\{t(m)\}$ *are sequences satisfying the conditions (1). If* $(D_{\beta,\gamma}) - \lim_{n,m\to\infty} x_{nm} = L$, *then*

$$(D_{\beta,\gamma})\, st_2 - \lim_{n,m\to\infty} x_{nm} = L.$$

Corollary

If $x_{nm} \to L(n,m \to \infty)$, *then* $(D_{\beta,\gamma})\, st_2 - \lim_{n,m\to\infty} x_{nm} = L.$

Remark:

Generally, the inverse of last theorem is not true. For example; let $q(n)$ and $r(m)$ are monotone increasing sequences of positive integers and h_1, h_2 are fixed numbers. Let we define a double sequence for $n, m = 1, 2, \ldots$,

$$x_{kl} := \begin{cases} k^2 l^2, & \begin{cases} \|\sqrt{q(n)}\| - h_1 < k \leq \|\sqrt{q(n)}\|, \\ \|\sqrt{r(m)}\| - h_2 < l \leq \|\sqrt{r(m)}\| \end{cases} \\ 0, & \text{otherwise.} \end{cases}$$

If we considerate $(D_{\beta,\gamma})$-method for the sequences $p(n), t(m)$ that the conditions

$$0 < p(n) \leq \|\sqrt{q(n)}\| - h_1,\ 0 < t(m) \leq \|\sqrt{r(m)}\| - h_2$$

are hold, we obtained $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$.
On the other hand, we have

$$\lim_{n,m\to\infty} \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),\ r(m)} |x_{kl} - L| \geq h_1 h_2.$$

Given that, since $h_1 \neq 0$ and $h_2 \neq 0$, $(D_{\beta,\gamma}) - \lim_{n,m\to\infty} x_{nm} \neq L$ is obtained.

Theorem

Let a double sequence $x = (x_{nm})$ be bounded. If $(D_{\beta,\gamma})$ $st_2 - \lim_{n,m\to\infty} x_{nm} = L$, then $(D_{\beta,\gamma}) - \lim_{n,m\to\infty} x_{nm} = L$.

Theorem

Let $x = (x_{nm})$ *be a double sequence such that* $x_{nm} \in \mathbb{R}$ *for all* $n,\ m \in \mathbb{N}$ *and* $L \in \mathbb{R}$. *If* $st_2 - \lim_{n,m\to\infty} x_{nm} = L$, *then* $(D_{\beta,\gamma})$ $st_2 - \lim_{n,m\to\infty} x_{nm} = L$.

Remark: The inverse of last theorem is not true. For instance; if we define a sequence

$$x_{nm} := \begin{cases} \frac{(n+1)\ (m+1)}{2}, & n \text{ and } m \text{ odd} \\ -\frac{n\ m}{2}, & n \text{ or } m \text{ even} \end{cases}$$

and choose $p(n) = 2n,\ q(n) = 4n,\ t(m) = 2m,\ r(m) = 4m$, then

$$(D_{2n,\ 2m}) - \lim_{n,m\to\infty} x_{nm} = 0.$$

Hence we get

$$(D_{2n,\ 2m})\ st_2 - \lim_{n,m\to\infty} x_{nm} = 0.$$

But, for all $\varepsilon > 0$,

$$\lim_{n,m\to\infty} \frac{\left|\{k \leq n,\ l \leq m : |x_{kl} - 0| \geq \varepsilon\}\right|}{n\,m} \neq 0$$

i.e., $st_2 - \lim_{n,m\to\infty} x_{nm} \neq 0$.

Corollary

Under the conditions last theorem, let $q(n),\ r(m)$ are sequences of positive integers such that $q(n) < n,\ r(m) < m$, for all $n, m \in \mathbb{N}$. If $st_2 - \lim_{n,m\to\infty} x_{nm} = L$, then $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$.

Theorem

Let $x = (x_{nm})$ *be a double sequence,* $q(n) = n$*,* $r(m) = m$ *for all* $n, m \in \mathbb{N}$ *and let* $\{p(n)\}, \{t(m)\}$ *be arbitrary two sequences.* $(D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$ *if and only if* $st_2 - \lim_{n,m\to\infty} x_{nm} = L$.

Corollary

Let $x = (x_{nm})$ *be a double sequence,* $\{q(n)\}$ *and* $\{r(m)\}$ *is equal to almost all positive integers. Then, for arbitrary sequences* $\{p(n)\}, \{t(m)\}, (D_{\beta,\gamma})\ st_2 - \lim_{n,m\to\infty} x_{nm} = L$ *implies* $st_2 - \lim_{n,m\to\infty} x_{nm} = L$

Theorem

Let $x = (x_{nm})$ *be a double sequence, let* $\{q(n)\}$ *and* $\{r(m)\}$ *be sequences of positive integers with* $p(n) = n-1$, $t(m) = m-1$. *In order that* $(D_{\beta,\gamma})$ $st_2 - \lim_{n,m\to\infty} x_{nm} = L$ *implies* $st_2 - \lim_{n,m\to\infty} x_{nm} = L$, *it is necessary and sufficient that the sequences* $\{q(n) - n\}$ *and* $\{r(m) - m\}$ *be bounded.*

Theorem

Let $x = (x_{nm})$ *be a double sequence. In order that* $(D_{\beta,\gamma})$ $st_2 - \lim_{n,m\to\infty} x_{nm} = L$ *implies* $st_2 - \lim_{n,m\to\infty} x_{nm} = L$ *where* $p(n) = n - 1,\ t(m) = m - 1,$

$$q(n) := \begin{cases} h_{i+1} - 1, & n = h_i,\ i = 1, 2, \ldots \\ n, & \textit{otherwise} \end{cases}$$

and

$$r(m) := \begin{cases} s_{j+1} - 1, & m = s_j,\ j = 1, 2, \ldots \\ m, & \textit{otherwise.} \end{cases}$$

$\{h_n\}, \{s_m\}$ *being increasing sequences of integers for which* $h_n > n,\ s_m > m$, *it is necessary and sufficient that the sequences* $\left\{\frac{q(n)}{n}\right\}$ *and* $\left\{\frac{r(m)}{m}\right\}$ *be bounded.*

Recall that let S be a nonempty set and let K be a subset of S. Then, the characteristic function of K is the function $\chi_K : S \to \{0, 1\}$ defined by

$$\chi_K(s) = \begin{cases} 1 & , \quad \text{if } s \in K \\ 0 & , \quad \text{if } s \notin K. \end{cases}$$

Let $A = (a_{jk}^{nm})$ be a four dimensional summability matrix and let a subset K of $\mathbb{N}^2$. The A-density of a subset K of $\mathbb{N}^2$ is defined as

$$\delta_A^{(2)}(K) := \lim_{n,m\to\infty} \sum_{(j,k)\in K} a_{nmjk},$$

if the limit exists and finite.

A double sequence $x = (x_{jk})$ is said to be A-statistically convergent to L, written $(A)\, st_2 - \lim_{n,m\to\infty} x_{nm} = L$, if $\delta_A^{(2)}(K_\varepsilon) = 0$ for every $\varepsilon > 0$, where

$$K_\varepsilon := \{\, k \leq n,\ l \leq m :\ |x_{kl} - L| \geqslant \varepsilon\} \text{ [13].}$$

We examine the concept of $C_{\lambda,\mu}$-statistically convergence and its relations to $D_{\lambda,\mu}$-statistically convergence. Now, we define $C_{\lambda,\mu}$-statistically convergence, note that if $A = C_{\lambda,\mu}$, then $(C_{\lambda,\mu})\, st_2 - \lim_{n,m\to\infty} x_{nm} = L$ if for every $\varepsilon > 0$,

$$\delta^{(2)}_{C_{\lambda,\mu}}(K_\varepsilon) = \lim_{n,m\to\infty} \frac{1}{\lambda(n)\mu(m)} \left(C_{\lambda,\mu} \cdot \chi_{K_\varepsilon}\right)_{nm}$$

$$= \lim_{n,m\to\infty} \frac{1}{\lambda(n)\mu(m)} \sum_{k=1,l=1}^{\lambda(n),\mu(m)} \chi_{K_\varepsilon}(k,l)$$

$$= \lim_{n,m\to\infty} \frac{1}{\lambda(n)\mu(m)} \left|\left\{k \le \lambda(n),\ l \le \mu(m) : |x_{kl} - L| \ge \varepsilon\right\}\right| = 0.$$

Here, the vertical bars indicate the number of elements in the set.

Let $\lambda = (\lambda(n))$ and $\mu = (\mu(m))$ are strictly increasing sequences of positive integers such that $\lambda(0) = 0$ and $\mu(0) = 0$. Then, $D_{\beta,\gamma}$-statistically convergence is defined as $D_{\lambda,\mu}$-statistically convergence taking

$$q(n) = \lambda(n),\ p(n) = \lambda\left(n-1\right),\ r(m) = \mu(m) \text{ and } t(m) = \mu(m-1).$$

It is denoted by $(D_{\lambda,\mu})\ st_2$.
The following theorem gives us relation between $C_{\lambda,\mu}$-statistically convergence and $D_{\lambda,\mu}$-statistically convergence.

Theorem

Let $\lambda = (\lambda(n))$ *and* $\mu = (\mu(m))$ *are strictly increasing sequences of positive integers such that* $\lambda(0) = 0$ *and* $\mu(0) = 0$. *If* $(D_{\lambda,\mu})$ $st_2 - \lim_{n,m\to\infty} x_{nm} = L$, *then* $(C_{\lambda,\mu})$ $st_2 - \lim_{n,m\to\infty} x_{nm} = L$.

Theorem

The method $D_{\beta,\gamma}$ *includes* $C_1 = (C, 1, 1)$ *if and only if the sequences* $\left\{\frac{p(n)}{\beta(n)}\right\}$ *and* $\left\{\frac{t(m)}{\gamma(m)}\right\}$ *are bounded.*

Theorem

Let $\{\lambda(n)\}$ *and* $\{\mu(m)\}$ *be infinite subset of* $\mathbb{N}$ *with* $\lambda(0) = 0$ *and* $\mu(0) = 0$*. Then* $C_{\lambda,\mu}$ *includes* $D_{\beta,\gamma}$*.*

Theorem

Let $\{\lambda(n)\}$ *and* $\{\mu(m)\}$ *be infinite subsets of* $\mathbb{N}$ *with* $\lambda(0) = 0$, $\mu(0) = 0$*. Then* $D_{\beta,\gamma}$ *includes* $C_{\lambda,\mu}$ *if and only if*

$$\liminf_{n\to\infty} \frac{\lambda(n)}{\lambda(n-1)} > 1 \quad \textit{and} \quad \liminf_{m\to\infty} \frac{\mu(m)}{\mu(m-1)} > 1.$$

Theorem

Let $\{\lambda(n)\}$ *and* $\{\mu(m)\}$ *be infinite subsets of* $\mathbb{N}$ *with* $\lambda(0) = 0$, $\mu(0) = 0$. *Then* $D_{\beta,\gamma}$ *is equivalent to* $C_{\lambda,\mu}$ *if and only if* $\liminf_{n\to\infty} \frac{\lambda(n)}{\lambda(n-1)} > 1$ *and* $\liminf_{m\to\infty} \frac{\mu(m)}{\mu(m-1)} > 1$.

[1] Agnew R,P., On Deferred Cesàro Means, Ann. of Math., 33, (1932), 413-421.
[2] Armitage D. H., Maddox I. J., A New Type of Cesàro Mean, Analysis 9, 195-206 (1989).
[3] Buck R. P., Generalized Asymptotic Density, Amer. J. Math. Comm., 75 (1953), 335-346.
[4] Connor J. S., The Statistical and Strong p-Cesàro Convergence of Sequences, Analysis, 8 (1988) 47-63.
[5] Demirci K., A-statistical core of a sequence, Math., 33 No:2 (2000), 343-353.

[6] Demirci K., On A-statistical Cluster Points, Glasnik Math., vol:37 57(2002), 293-301.
[7] Fast H., Sur La Convergence Statistique, Colloq. Math., 2, (1951), 241-244.
[8] Fridy J. A., On Statistical Convergence, Analysis 5 (1985), 301-313.
[9] Goffman C., Petersen G. M., Submethods of Regular Matrix Summability Methods, Canad. J. Math. 8, 40-46 (1956).
[10] Hamilton J. A., Transformations of Multiple Sequences, Duke Math. J. 2,29-60 (1936).

[11] Mursaleen, M and Edely ,Osama H.H., Statistical Convergence of Double sequences, J.Math. Anal. Appl. 288(2003)223-231.
[12] Mursaleen M., Mohiuddine S. A., Convergence Methods for Double Sequences and Applications, Springer (2014).
[13] H. I. Miller, A-statistical Convergence of Subsequences of Double Sequences, Bullettino U.M.I. (8), 10-B, (2007) 727-739.
[14] Osikiewicz J. A., Equivalence Results for Cesàro Submethods, Analysis 20, 35-43 (2000).
[15] Pringsheim A., Elementare Theorie der Unendliche Doppelreihen, Sitsungs Berichte der Math. Akad. der Wissenschafften zu Müch. Ber. 7, 101-153 (1898).

[16] Robison G. M. , Divergent Double sequences and series, Transactions of the American Mathematical Society, 28, no .1, (1926), 50-73.
[17] Šalát T., On Statistically Convergent Sequences of Real Numbers, Math. Slovaca 30 (1980), 139-150.
[18] Schoenberg I. J., The integrability of Certain Functions and Related Summability Methods, Amer. Math. Monthly, 66, (1959), 361-375.
[19] Ünver M., Inclusion results for four dimensional Cesàro Submethods, Stud. Univ. Babeş-Bolyai Math. 58 no:1 (2013) 43-54.

THANK YOU FOR LISTENING