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A NOTE ON LAGUERRE MATRIX POLYNOMIALS

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Abstract. In this paper, some new relations for Laguerre matrix polynomials are given.

1. Introduction

Recently, matrix polynomials that are solutions of a second order matrix differential equation are very popular subject in mathematics. In this area, many papers have been published ([17],[16],[18],[11],[19],[10],[23]). Many properties, extensions and generalizations of them have been introduced ([20],[9],[13],[12],[22],[15],[4],[24],[2],[3],[1],[8]). Laguerre matrix polynomial is one of them ([21],[20],[7],[5],[6]).

In this paper, firstly a few lemmas are given. After, some new relations for Laguerre matrix polynomials are obtained by using these lemmas.

Throughout this paper, for a matrix $A \in \mathbb{C}^{r\times r}$, $\sigma(A)$ denotes the set of all eigenvalues of A and is called its spectrum. A is a positive stable matrix if $\mathrm{Re}\,(\lambda) > 0$ for all $\lambda \in \sigma(A)$. Furthermore, the identity matrix and the null matrix in $\mathbb{C}^{r\times r}$ will be denoted I and $\mathbf{0}$. If $A_0, A_1, ..., A_n$ are elements of $\mathbb{C}^{r\times r}$ and $A_n \neq \mathbf{0}$, then

$$P(x) = A_n x^n + A_{n-1}x^{n-1} + ... + A_1 x + A_0$$

is called a matrix polynomial of degree n in x for every integer $n \geq 0$. From [17],

$$(A)_n = A(A+I)(A+2I)...(A+(n-1)I)\ ;\ n \geq 1\ ;\ (A)_0 = I. \tag{1.1}$$

is written. Using (1.1), we see that

$$\frac{I}{(n-k)!} = (-1)^k \frac{(-nI)_k}{n!}\ ;\ 0 \leq k \leq n. \tag{1.2}$$

In [17], if $f(z)$ and $g(z)$ are holomorphic functions which are defined in an open set Ω of the complex plane, and if A is a matrix in $\mathbb{C}^{r\times r}$ for which $\sigma(A) \subset \Omega$, using the properties of the matrix functional calculus in [14] then

$$f(A)g(A) = g(A)f(A).$$

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Hence, if $B \in \mathbb{C}^{r\times r}$ is a matrix for which $\sigma(B) \subset \Omega$ and $AB = BA$, then

$$f(A)g(B) = g(B)f(A).$$

Let A be a matrix in $\mathbb{C}^{r\times r}$satisfying $(-k) \notin \sigma(A)$ for every integer $k > 0$ and λ be a complex number whose real part is positive. In [17], n-th degree Laguerre matrix polynomial, $L_n^{(A,\lambda)}(x)$ is defined by

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\,k!}\,(A+I)_n\,(A+I)_k^{-1}\,(\lambda x)^k\,.$$

By using (1.2), $L_n^{(A,\lambda)}(x)$ can be written in the form

$$L_n^{(A,\lambda)}(x) = \frac{(A+I)_n}{n!}\sum_{k=0}^{n}(-nI)_k\,(A+I)_k^{-1}\,\frac{(\lambda x)^k}{k!}.$$

Also, Laguerre matrix polynomials have the following derivative relation [20],

$$\frac{d}{dx}L_n^{(A,\lambda)}(x) = -\lambda L_{n-1}^{(A+I,\lambda)}(x) \quad , \quad n \geq 1. \tag{1.3}$$

Lemma 1.1. [7] *The raising operator for Laguerre matrix polynomials is*

$$\frac{d}{dx}\left(x^A e^{-\lambda x}L_n^{(A,\lambda)}(x)\right) = (n+1)\,x^{A-I}e^{-\lambda x}L_{n+1}^{(A-I,\lambda)}(x)\,, \quad x > 0 \tag{1.4}$$

where A is positive stable matrix in $\mathbb{C}^{r\times r}$and $\operatorname{Re}(\lambda) > 0$.

2. Some new relations for Laguerre matrix polynomials

Lemma 2.1. *Let A be a matrix in $\mathbb{C}^{r\times r}$satisfying the spectral condition*

$$\operatorname{Re}(\mu) > 1 \ \text{ for all } \ \mu \in \sigma(A), \tag{2.1}$$

and λ be a complex number with $\operatorname{Re}(\lambda) > 0$. For Laguerre matrix polynomials,

$$\frac{d}{dx}\left[x^A L_n^{(A,\lambda)}(x)\right] = (A+nI)\,x^{A-I}L_n^{(A-I,\lambda)}(x) \quad , \quad x > 0 \tag{2.2}$$

is satisfied.

Proof. We start by taking the derivative of $x^A L_n^{(A,\lambda)}(x)$ with respect to x. Thus, we have

$$\begin{aligned}
\frac{d}{dx}\left[x^A L_n^{(A,\lambda)}(x)\right] &= \frac{d}{dx}\left[x^A \frac{(A+I)_n}{n!}\sum_{k=0}^{n}(-nI)_k\,(A+I)_k^{-1}\frac{(\lambda x)^k}{k!}\right] \\
&= \frac{d}{dx}\left[\frac{1}{n!}\sum_{k=0}^{n}\lambda^k(-nI)_k\,(A+I)_n\,(A+I)_k^{-1}\frac{x^{A+kI}}{k!}\right] \\
&= \frac{1}{n!}\sum_{k=0}^{n}\left\{\lambda^k(-nI)_k\,(A+I)_n\,(A+I)_k^{-1}\,(A+kI)\,\frac{x^{A+(k-1)I}}{k!}\right\}.
\end{aligned}$$

Then by using (1.1),

$$\begin{aligned}\frac{d}{dx}\left[x^{A}L_{n}^{(A,\lambda)}(x)\right] &= \frac{x^{A-I}}{n!}\sum_{k=0}^{n}\lambda^{k}(-nI)_{k}(A)_{n}(A+nI)(A)_{k}^{-1}\frac{x^{k}}{k!}\\ &= (A+nI)\,x^{A-I}\frac{(A)_{n}}{n!}\sum_{k=0}^{n}(-nI)_{k}(A)_{k}^{-1}\frac{(\lambda x)^{k}}{k!}\\ &= (A+nI)x^{A-I}L_{n}^{(A-I,\lambda)}(x).\end{aligned}$$

holds. This completes the proof. □

Theorem 2.1. *Let A be a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition* (2.1) *and* $\operatorname{Re}(\lambda)>0$. *Laguerre matrix polynomials satisfy the following relation with* $x>0$

$$AL_{n}^{(A,\lambda)}(x)=(A+nI)L_{n}^{(A-I,\lambda)}(x)+\lambda xL_{n-1}^{(A+I,\lambda)}(x)\ ,\quad n\geq 1.$$

Proof. The derivative of multiplication of $x^{A}L_{n}^{(A,\lambda)}(x)$ with respect to x is, as follows from (1.3),

$$\begin{aligned}\frac{d}{dx}\left[x^{A}L_{n}^{(A,\lambda)}(x)\right] &= Ax^{A-I}L_{n}^{(A,\lambda)}(x)+x^{A}\frac{d}{dx}L_{n}^{(A,\lambda)}(x)\\ &= Ax^{A-I}L_{n}^{(A,\lambda)}(x)-\lambda x^{A}L_{n-1}^{(A+I,\lambda)}(x)\ ,\quad n\geq 1.\end{aligned}$$

Using (2.2) in the left side of this equation,

$$Ax^{A-I}L_{n}^{(A,\lambda)}(x)=(A+nI)\,x^{A-I}L_{n}^{(A-I,\lambda)}(x)+\lambda x^{A}L_{n-1}^{(A+I,\lambda)}(x)\ ,\quad n\geq 1$$

is written. Then multiplying both sides with the inverse of x^{A-I},

$$AL_{n}^{(A,\lambda)}(x)=(A+nI)L_{n}^{(A-I,\lambda)}(x)+\lambda xL_{n-1}^{(A+I,\lambda)}(x)\ ,\quad n\geq 1$$

is obtained. □

Theorem 2.2. *Let A be a matrix in $\mathbb{C}^{r\times r}$ satisfying the spectral condition* (2.1) *and* $\operatorname{Re}(\lambda)>0$. *For Laguerre matrix polynomials,*

$$(n+1)\,L_{n+1}^{(A-I,\lambda)}(x)=(A+nI)\,L_{n}^{(A-I,\lambda)}(x)-\lambda xL_{n}^{(A,\lambda)}(x)\ ,\quad x>0$$

holds.

Proof. Starting from the derivative of $e^{-\lambda x}x^{A}L_{n}^{(A,\lambda)}(x)$ with respect to x and using Lemma 2.1, we can write

$$\begin{aligned}\frac{d}{dx}\left[e^{-\lambda x}\left(x^{A}L_{n}^{(A,\lambda)}(x)\right)\right] &= -\lambda e^{-\lambda x}x^{A}L_{n}^{(A,\lambda)}(x)+e^{-\lambda x}\frac{d}{dx}\left(x^{A}L_{n}^{(A,\lambda)}(x)\right)\\ &= -\lambda e^{-\lambda x}x^{A}L_{n}^{(A,\lambda)}(x)+(A+nI)\,e^{-\lambda x}x^{A-I}L_{n}^{(A-I,\lambda)}(x).\end{aligned}\tag{2.3}$$

Combining (1.4) and (2.3), then multiplying both side with $e^{\lambda x}x^{-A+I}$, the proof is completed. □

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